Application of Hertzian Theory to Torus on Plane Contacts

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Hertzian theory includes a well-known analytical solution for the calculation of contact area and pressure between two bodies. Hertzian theory is not unconditionally applicable, in particular with regards to the shape of the contacting solids. When geometric assumptions are invalid, the finite element method is generally used, but is more computationally intensive. We propose a generalization to the geometric assumptions of Hertzian theory and apply it to torus on plane contacts. In order to evaluate the accuracy of the calculation, a finite element model is used as a basis for comparison.

Keywords: contact mechanics, Hertzian theory, general shape contacts, torus on plane contacts

1. Introduction

Hertzian theory is well-understood in modern day engineering, and there are several texts that provide derivations, for example [1]. Following the derivations, it is at first unclear why the results of classical Hertzian theory are unsatisfactory for torus on plane contacts, though some theories have been proposed [2]. In [2], the observation that the curvature is highly variable along the Cartesian tangent plane coordinates was made. We introduce a curvilinear, non-orthogonal coordinate system for the tangent plane in order to assign the curvatures along the principal curvature curves of the torus. In this coordinate system, a good parabolic approximation of the torus is possible, i.e. Hertzian theory can be generalized to this setting. We additionally propose methods for the change of coordinates calculation that enable practical computation.

2. Theory

One of the first steps in Hertzian theory is the Taylor series approximation of the gap height h using the principal curvatures ρ_x, ρ_y of the gap and choosing the coordinate system to be along the principal curvature directions [1]:

$$h \approx \frac{x^2}{2}\rho_x + \frac{y^2}{2}\rho_y \qquad (1)$$

Missing from this approximation are polynomial terms of higher order, some of which can also be described using geometric properties of the surface [3]. In the case of torus on plane contacts, the terms of higher order give information crucial for good approximation of the surface, specifically terms that describe the change of direction of the principal curvature directions when viewed from the tangent plane. A better approximation can thus be achieved either by including higher order terms in the surface approximation (even using the true surface), or by accounting for the missing information in a different way. Including terms of higher order has the two downsides that it is possible for the surface approximation to pass through the tangent plane (i.e. intersect the tangent plane other than at a single point) and the treatment of the pressure distribution is still unclear, necessitating a finite element calculation.

We therefore choose to account for missing information using a change of coordinates. Specifically, the change of coordinates takes into account that, unlike conventional Hertzian contacting solids, the geodesics of a torus are not the principal curvature curves. Using a parabolic surface approximation without a change of coordinates effectively assigns the principal curvatures to occur along the geodesics in the direction of the principal curvature directions. In general, these geodesics may only intersect the principal curvature curves at the coordinate system origin. With an appropriate change of coordinates, the principal curvatures can be designated correctly to lie along the true principal curvature curves of the surface.

For a torus, the correct construction of the new coordinate system is relatively clear: As a surface of revolution, the torus can be parametrized along its principal curvature directions. Using the normal projection onto the tangent plane of the principal curvature curves as a coordinate system therefore takes into account the missing information.

3. Methods

There are two models that require verification: The approximation of the surface using the change of coordinates and the Hertzian pressure calculation in the new coordinate system.

In order to verify the surface approximation, a refined brute force search algorithm was used to reconstruct the exact surface as well as the exact surface parametric coordinates. The exact parametric coordinates are used to verify the change of coordinates formula as well as the parabolic approximation of the surface.

In order to verify the resulting pressure distribution, finite element simulations were done, varying the torus dimensions as well as the angle of contact between the torus and the plane.

4. Acknowledgments

The authors are thankful for the support of DFG Grant number 407531729.

5. References

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